COMP 545: Advanced topics in optimization
From simple to complex ML systems

Lecture 3

Figure 1

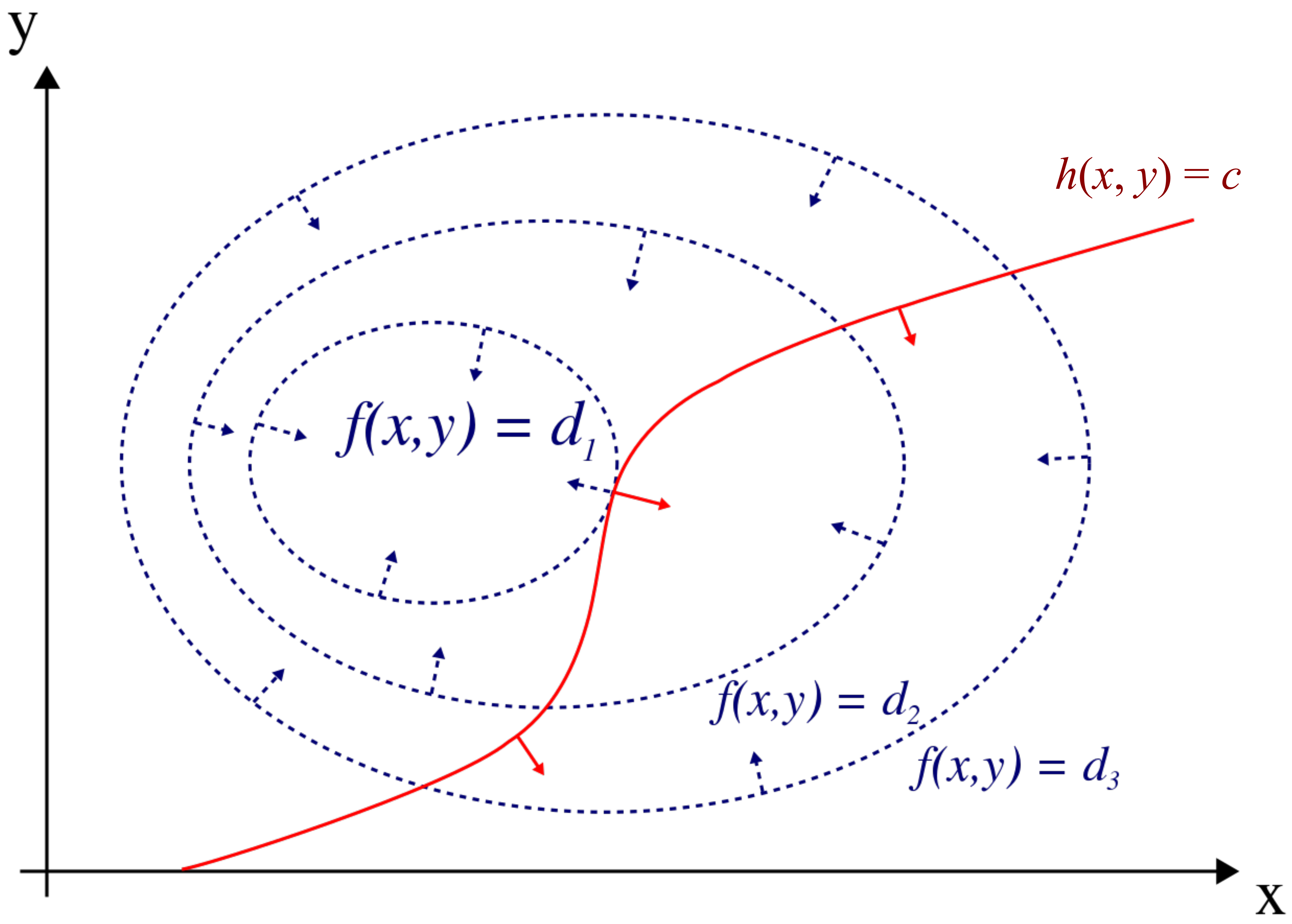
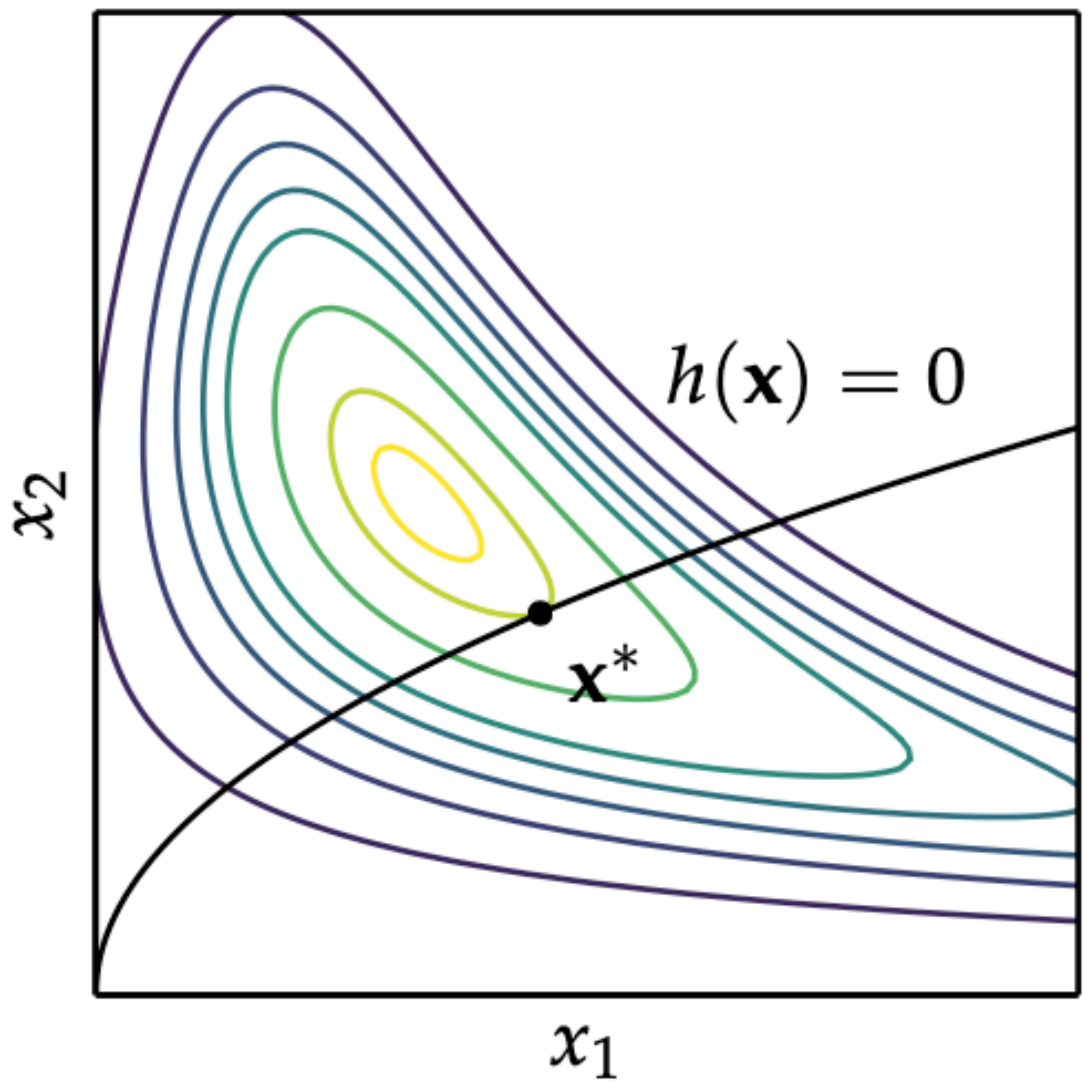


Figure 1: The red curve shows the constraint $h(x, y) = c$. The blue curves are contours of $f(x, y)$. The point where the red constraint tangentially touches a blue contour is the maximum of $f(x, y)$ along the constraint, since $d_1 > d_2$.

Figure 2



Example with multiple equality constraints

Objective:

$$f(x) = x_1^2 + x_2^2 + x_3^2$$

Constraints:

$$h_1(x) = 0 \Rightarrow x_1 + x_2 - 2 = 0$$
$$h_2(x) = 0 \Rightarrow x_1 + x_3 - 2 = 0$$

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Form the Lagrangian:

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and solve:

$$0 = \nabla_{x_1}\mathcal{L}(x, \lambda_1, \lambda_2) = \nabla_{x_2}\mathcal{L}(x, \lambda_1, \lambda_2) = \nabla_{x_3}\mathcal{L}(x, \lambda_1, \lambda_2) = \nabla_{\lambda_1}\mathcal{L}(x, \lambda_1, \lambda_2) = \nabla_{\lambda_2}\mathcal{L}(x, \lambda_1, \lambda_2)$$

Example with multiple equality constraints

Closed form solution:

$$\lambda_1 = -\frac{4}{3}$$
$$\lambda_2 = -\frac{4}{3}$$
$$x_1 = \frac{4}{3}$$
$$x_2 = \frac{2}{3}$$
$$x_3 = \frac{2}{3}$$

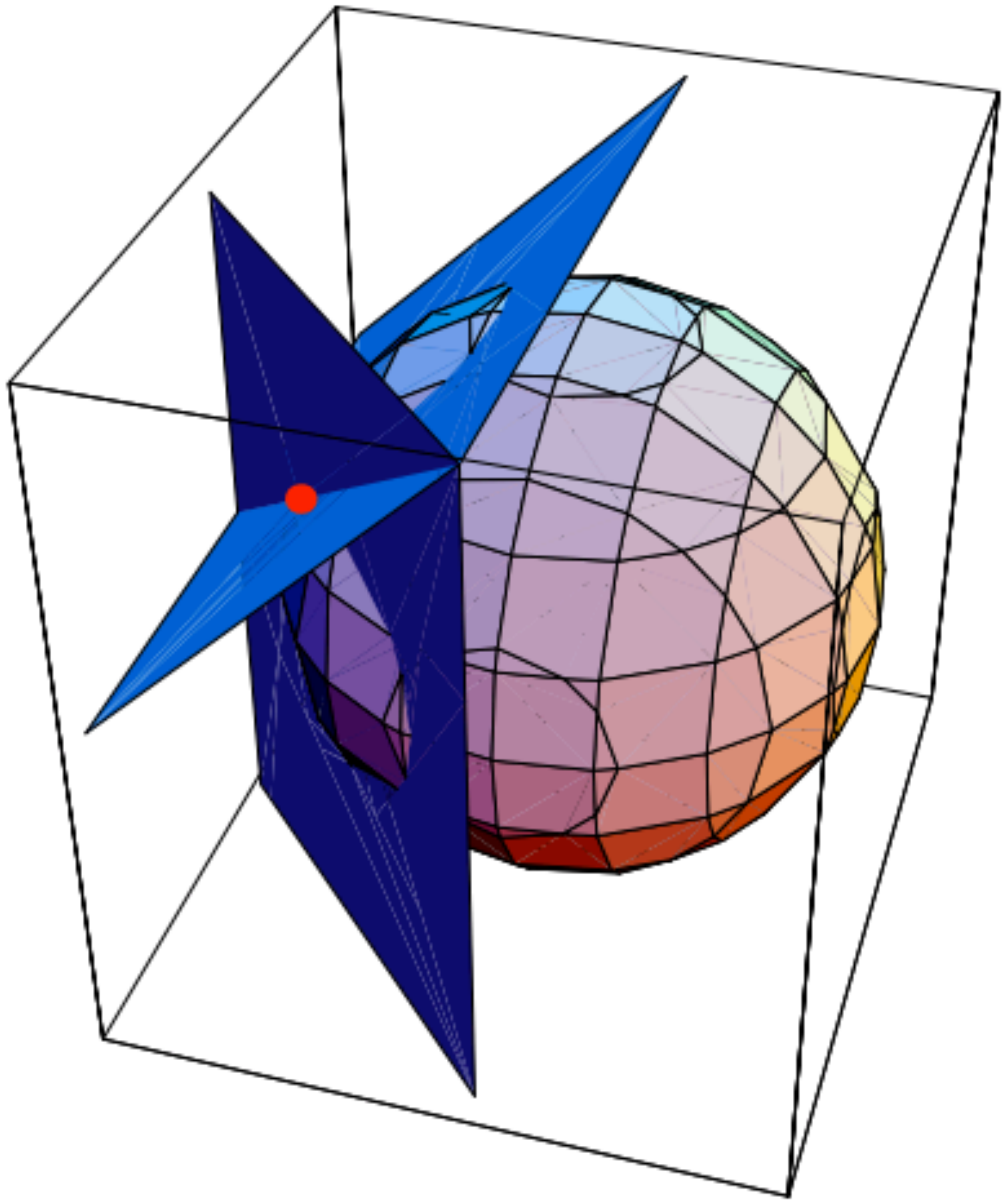
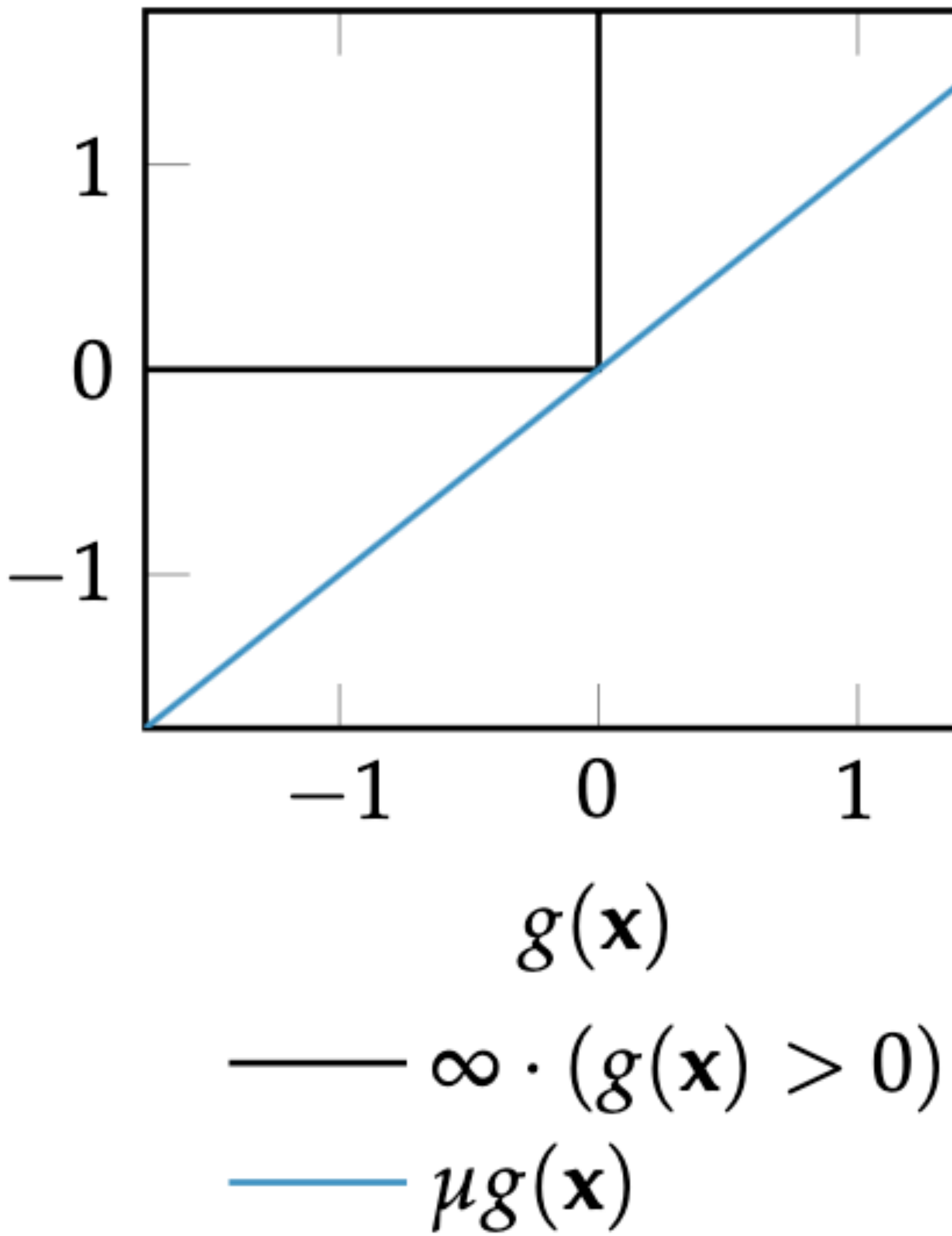


Figure 3



Example of general Lagrangian function: Robotics

("Motion Planning with Sequential Convex Optimization and Convex Collision Checking")

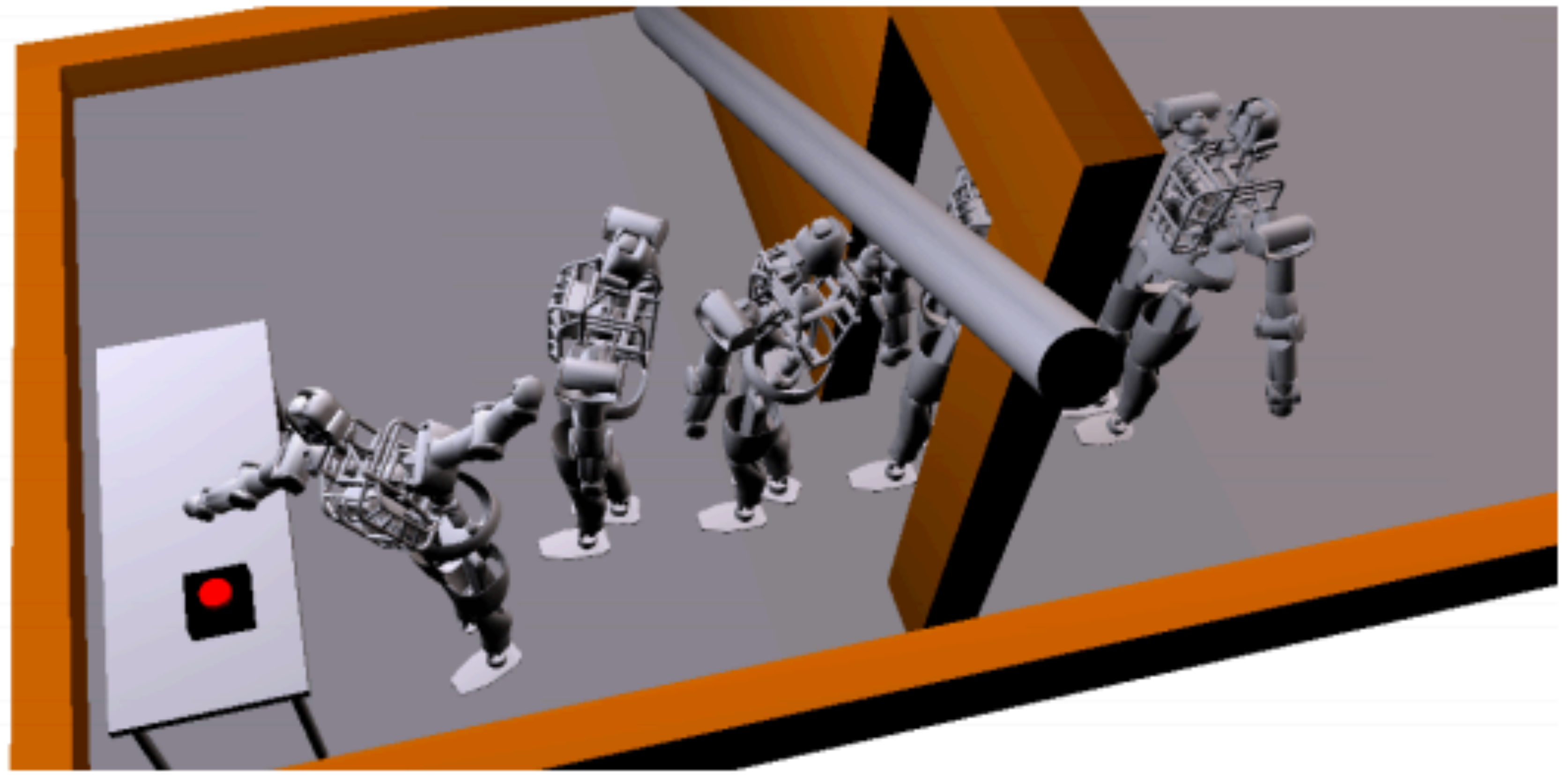


Fig. 8. The Atlas humanoid robot in simulation walking across the room while avoiding the door frame and other obstacles in the environment, and pushing a button. Each footstep was planned for separately using TrajOpt while maintaining static stability. Five time steps of the trajectory are shown.

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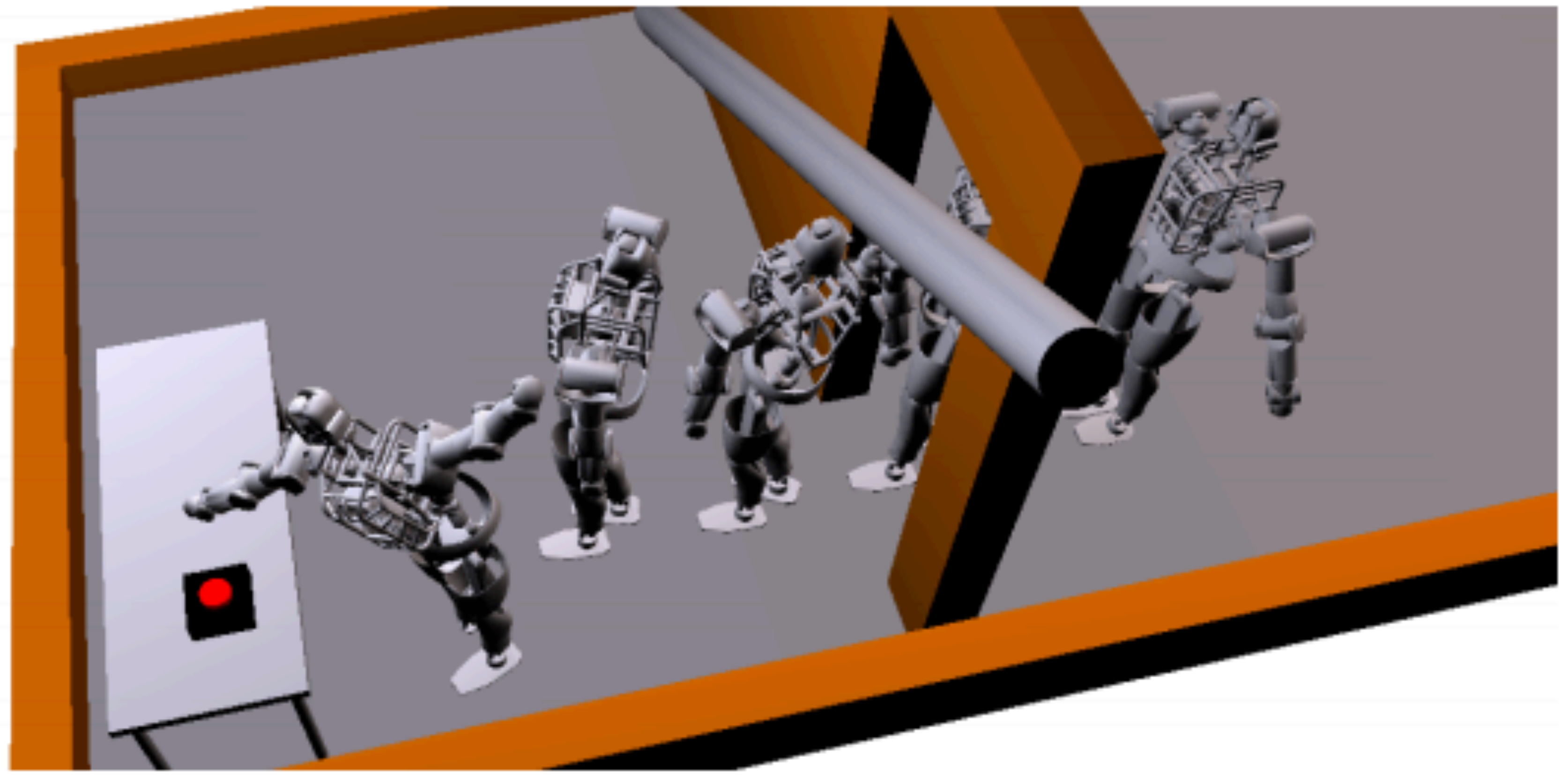

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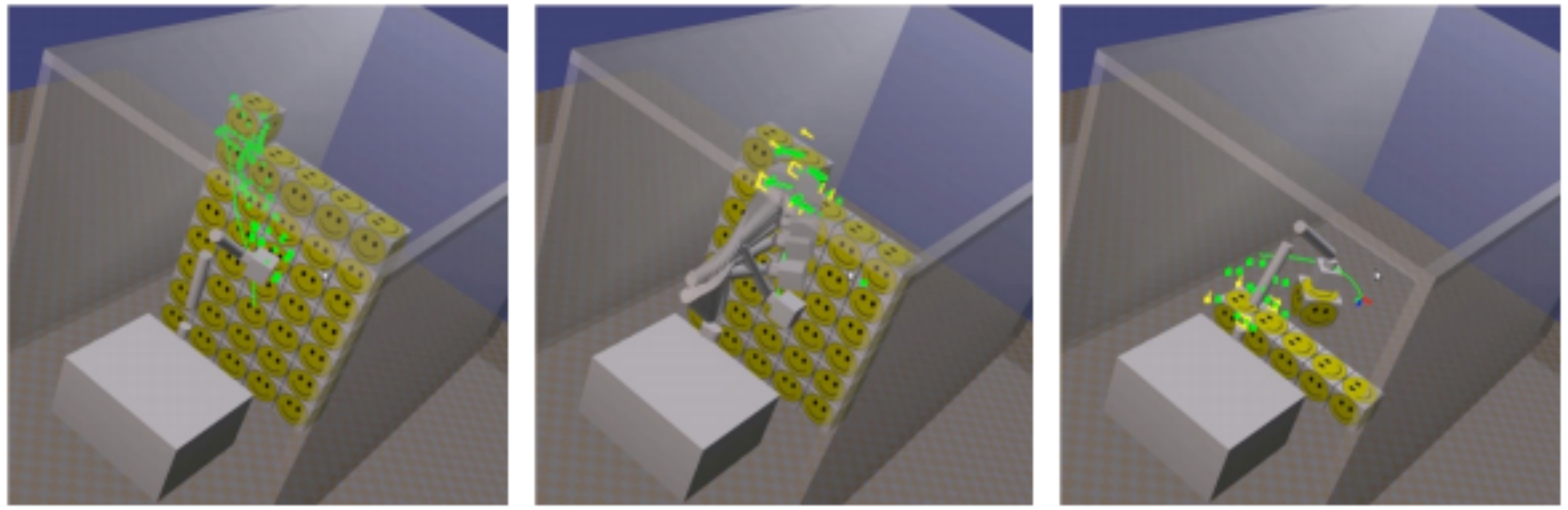

Fig. 9. Several stages of a box picking procedure, in which boxes are taken from the stack and moved to the side. The box, and hence the end effector of the robot arm, is subject to pose constraints.

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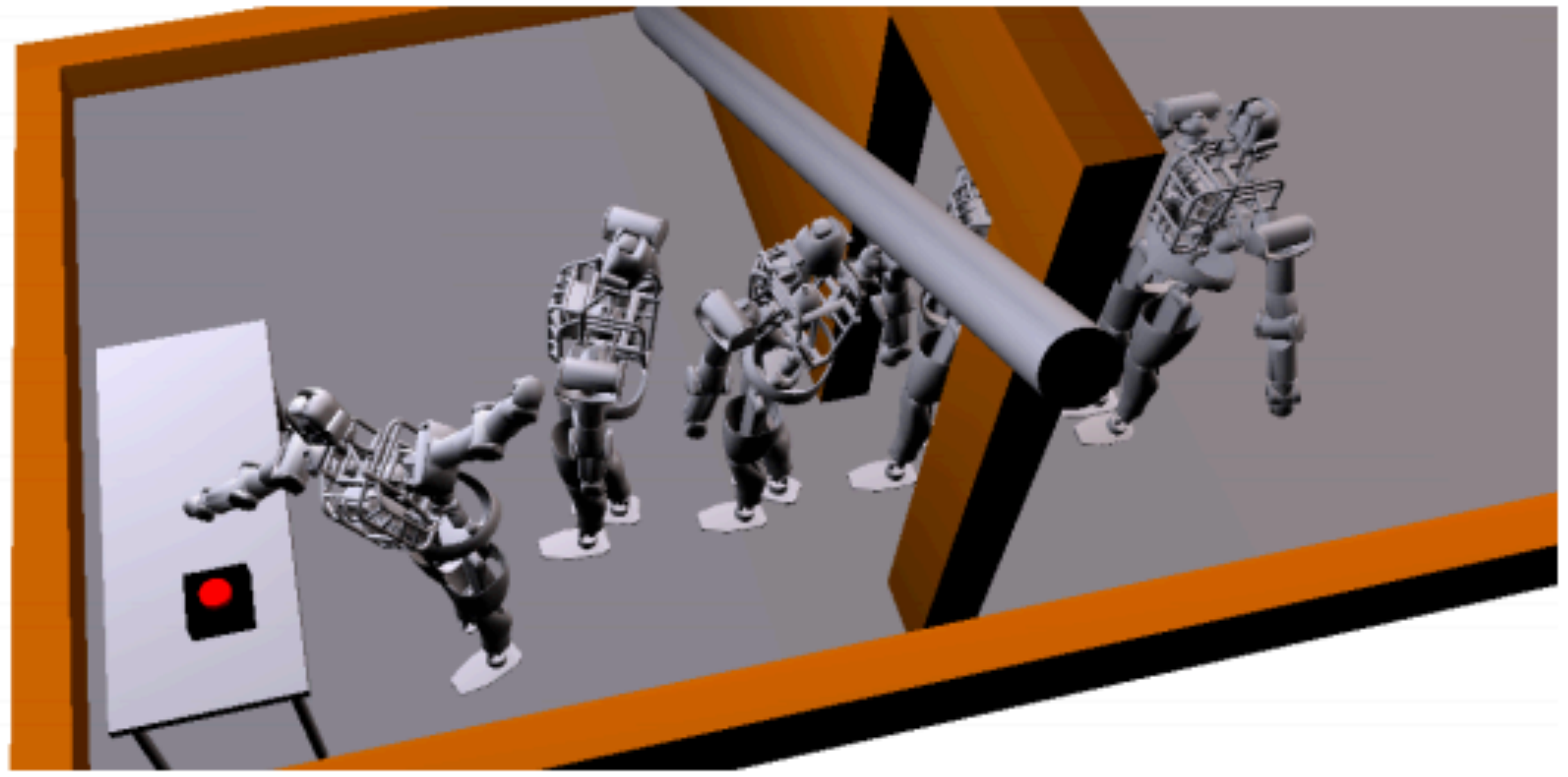

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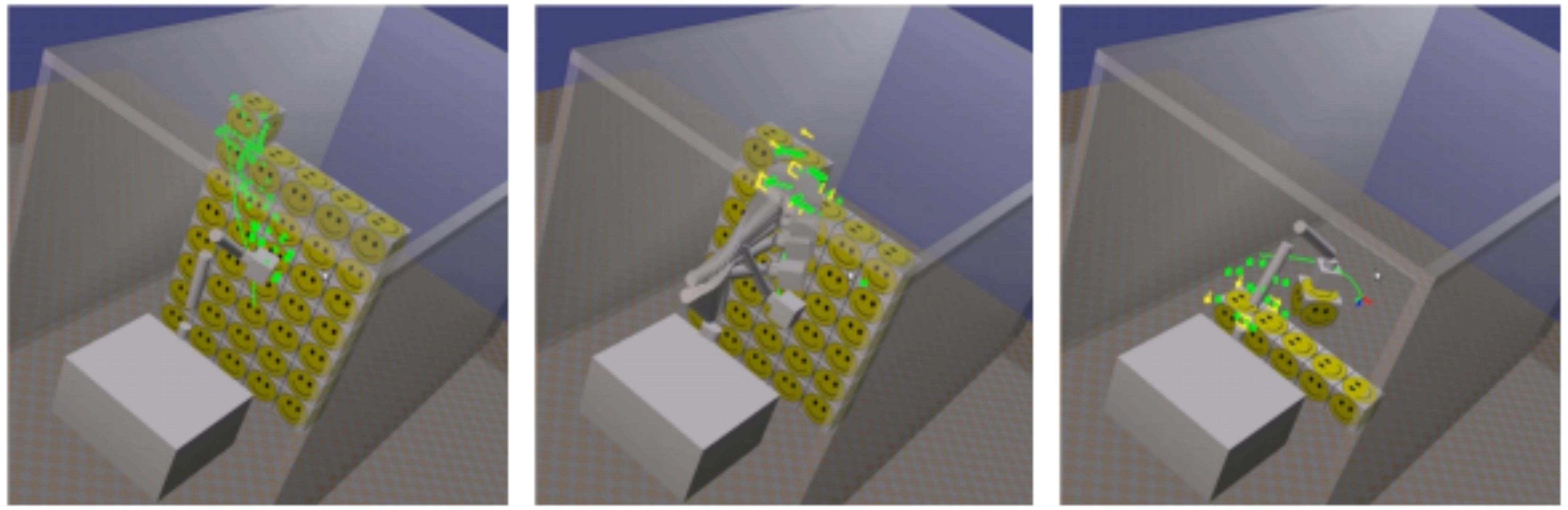

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Curvature-constrained planning problem in 3D environments as a nonlinear, constrained optimization problem:

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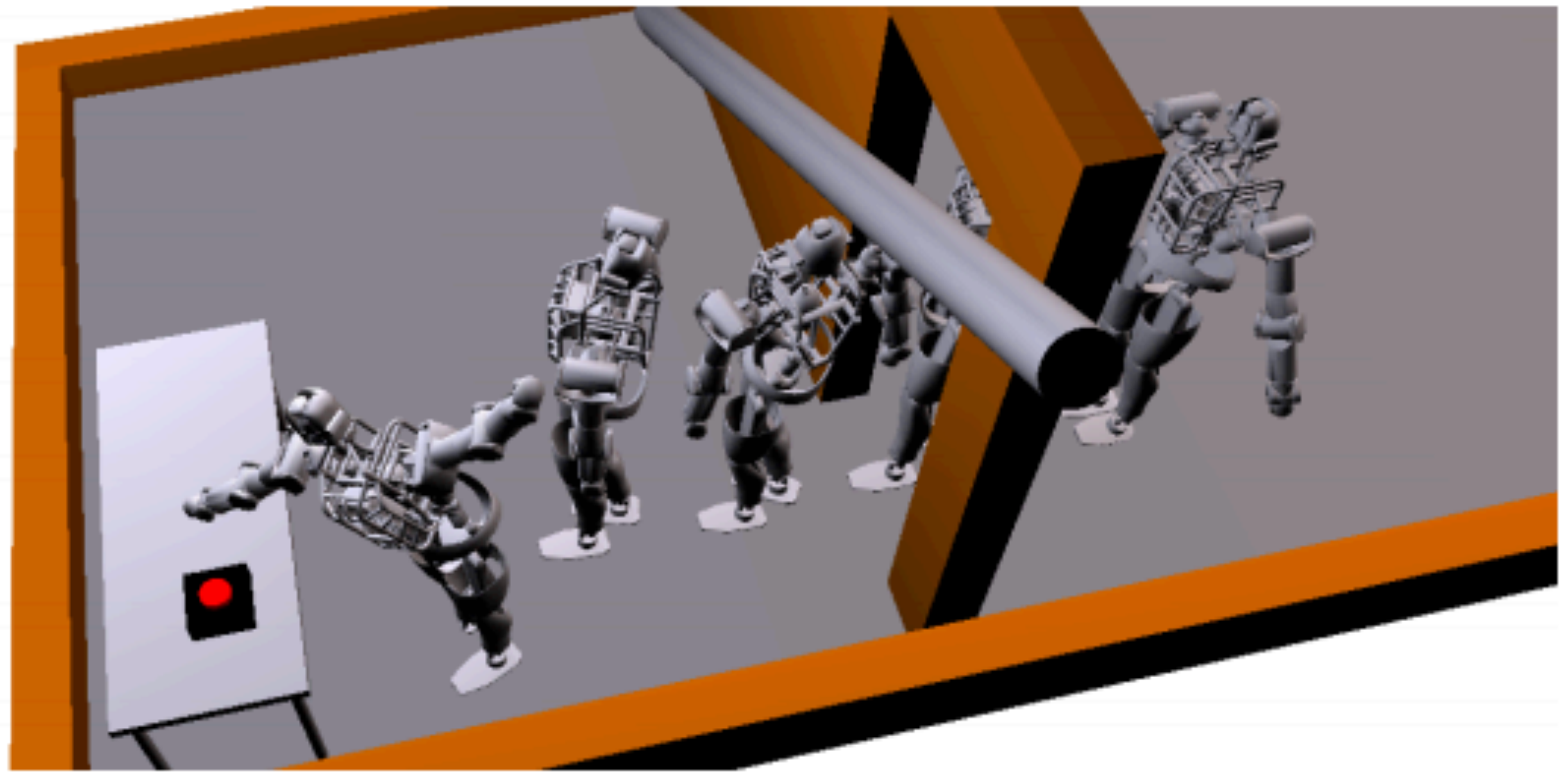

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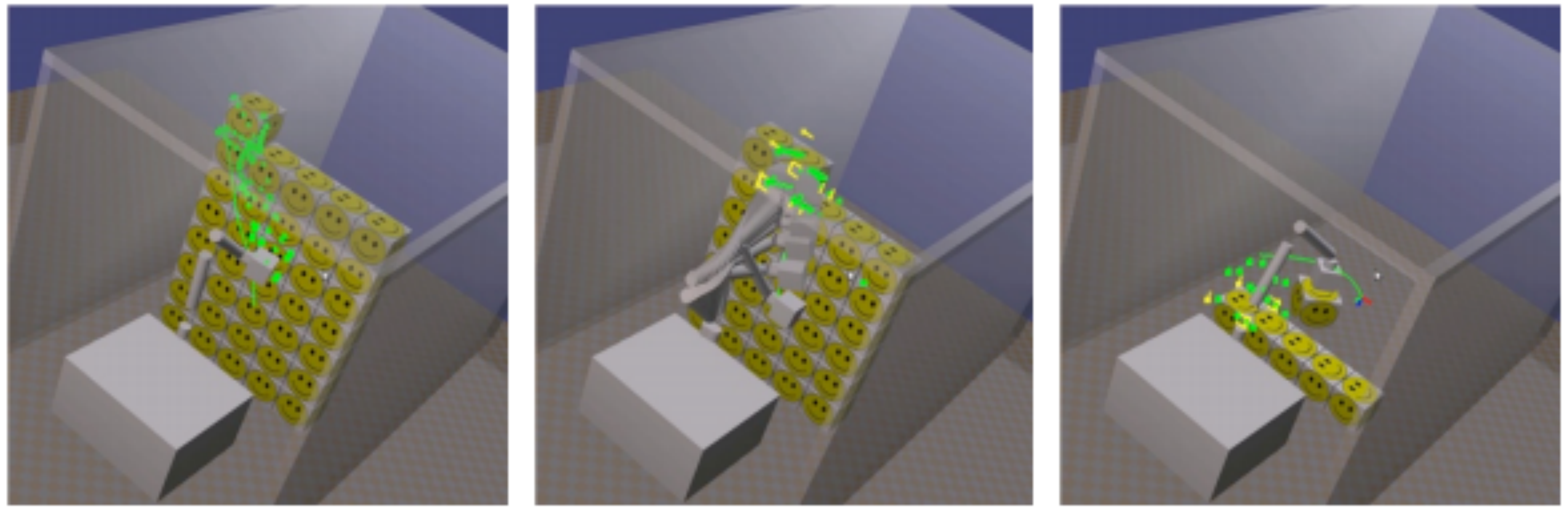

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Curvature-constrained planning problem in 3D environments as a nonlinear, constrained optimization problem:

$$\min_{\mathcal{X},\mathcal{U}} \ \alpha_\Delta \mathrm{Cost}_\Delta + \alpha_\phi \mathrm{Cost}_\phi + \alpha_\mathcal{O} \mathrm{Cost}_\mathcal{O}, \tag{33a}$$

$$\text{s.t.} \quad \log((X_t \cdot \exp(\mathbf{w}_t^\wedge) \cdot \exp(\mathbf{v}_t^\wedge))^{-1} \cdot X_{t+1})^\vee = \mathbf{0}_6, \tag{33b}$$

$$\mathrm{sd}(X_t, X_{t+1}, \mathcal{O}_i) \geq d_{\text{safe}} + d_{\text{arc}}, \tag{33c}$$

$$X_0 \in \mathcal{P}_{\text{entry}}, \ X_T \in \mathcal{P}_{\text{target}}, \tag{33d}$$

$$-\pi \leq \phi_t \leq \pi, \tag{33e}$$

$$\kappa_t = \kappa_{\max} \quad \text{or} \quad 0 \leq \kappa_t \leq \kappa_{\max}, \tag{33f}$$

$$\Delta \sum_{t=0}^{T-1} \kappa_t \leq c_{\max} \quad \text{for channel planning}, \tag{33g}$$

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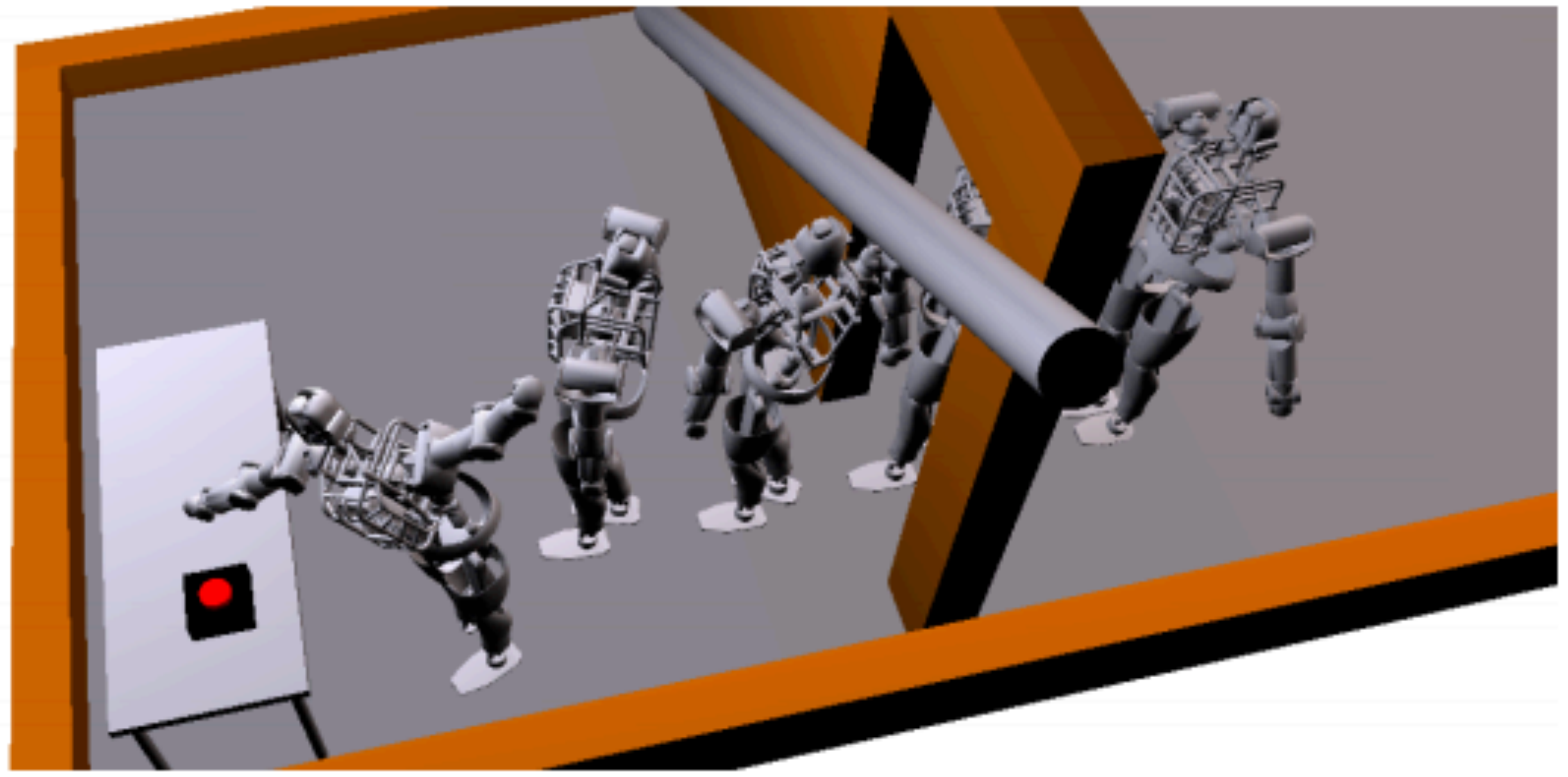

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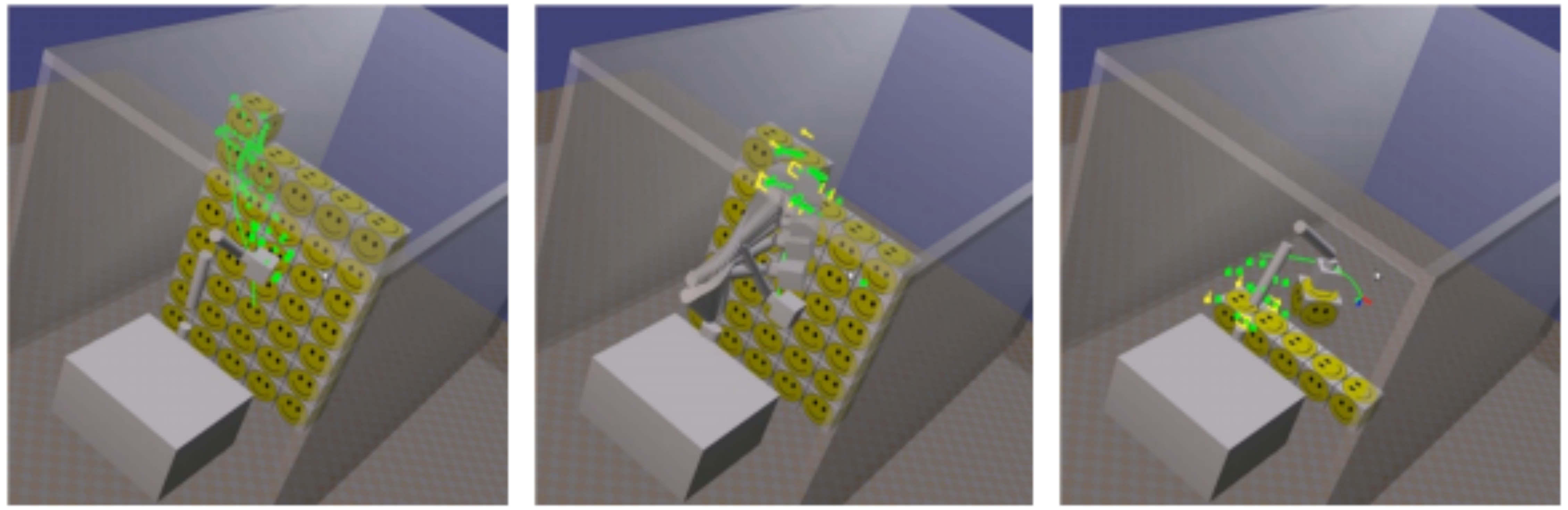

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Curvature-constrained planning problem in 3D environments as a nonlinear, constrained optimization problem:

$$\begin{aligned}&\text{minimize } f(\mathbf{x})\\&\text{subject to}\\&\quad g_i(\mathbf{x}) \leq 0, \quad i = 1, 2, \ldots, n_{ineq}\\&\quad h_i(\mathbf{x}) = 0, \quad i = 1, 2, \ldots, n_{eq}\end{aligned}$$

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("Motion Planning with Sequential Convex Optimization and Convex Collision Checking")

1: **for** PenaltyIteration $= 1, 2, \ldots$ **do**
2:     **for** ConvexifyIteration $= 1, 2, \ldots$ **do**
3:         $\tilde{f}, \tilde{g}, \tilde{h} = \text{ConvexifyProblem}(f, g, h)$
4:         **for** TrustRegionIteration $= 1, 2, \ldots$ **do**
5:             $\mathbf{x} \leftarrow \arg\min_{\mathbf{x}} \tilde{f}(\mathbf{x}) + \mu \sum_{i=1}^{n_{ineq}} |\tilde{g}_i(\mathbf{x})|^+ + \mu \sum_{i=1}^{n_{eq}} |\tilde{h}_i(\mathbf{x})|$
                subject to trust region and linear constraints
6:             **if** TrueImprove / ModelImprove $> c$ **then**
7:                 $s \leftarrow \tau^+ * s$ ▷ Expand trust region
8:                 **break**
9:             **else**
10:                 $s \leftarrow \tau^- * s$ ▷ Shrink trust region
11:             **if** $s <$ xtol **then**
12:                 **goto** 15
13:         **if** converged according to tolerances xtol or ftol **then**
14:             **break**
15:     **if** constraints satisfied to tolerance ctol **then**
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17:     **else**
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Figure 4

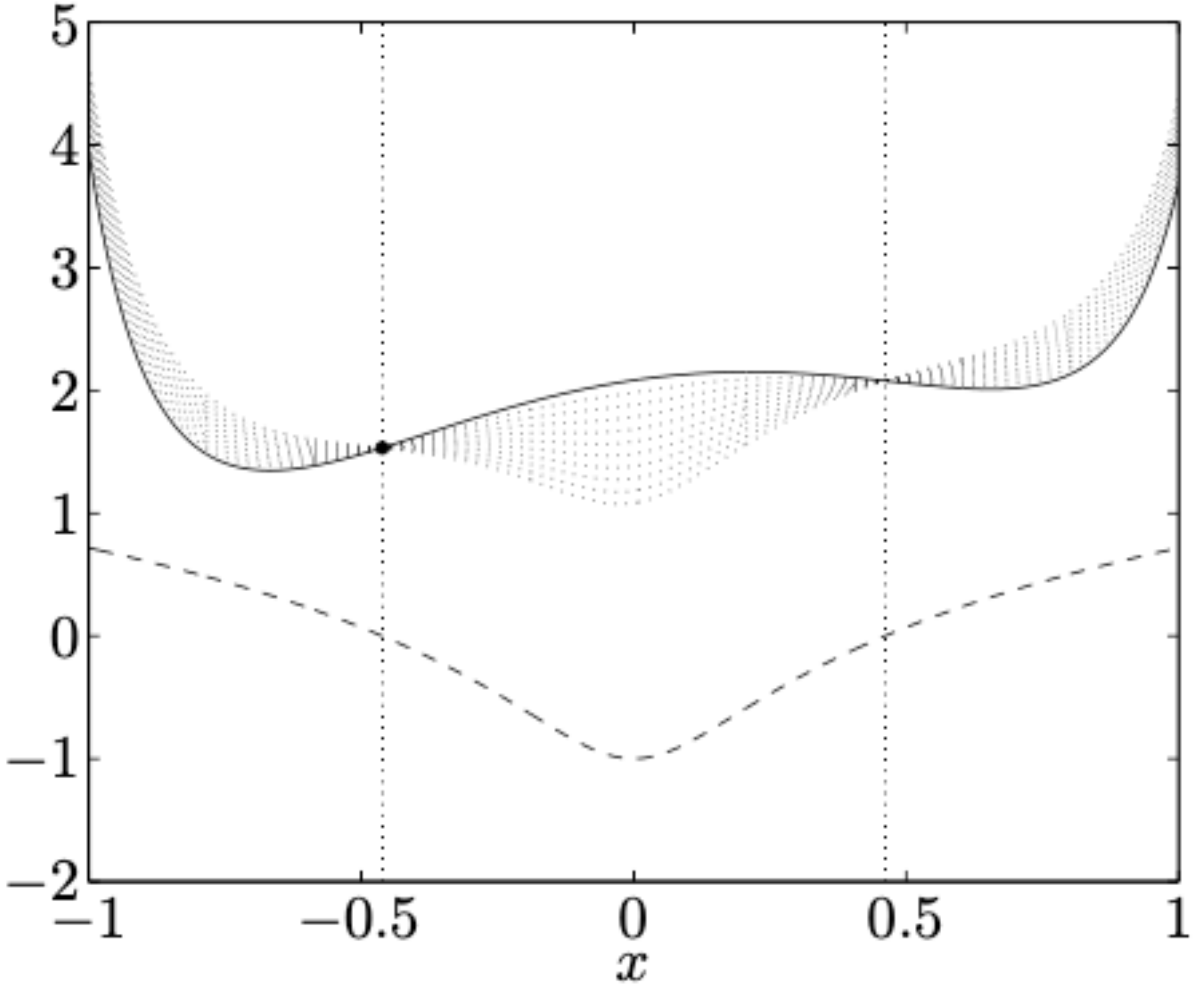

Figure 5.1 *Lower bound from a dual feasible point.* The solid curve shows the objective function f_0, and the dashed curve shows the constraint function f_1. The feasible set is the interval $[-0.46, 0.46]$, which is indicated by the two dotted vertical lines. The optimal point and value are $x^\star = -0.46$, $p^\star = 1.54$ (shown as a circle). The dotted curves show $L(x, \lambda)$ for $\lambda = 0.1,\ 0.2, \ldots, 1.0$. Each of these has a minimum value smaller than $p^\star$, since on the feasible set (and for $\lambda \geq 0$) we have $L(x, \lambda) \leq f_0(x)$.